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CONFORMALITY AND SEMICONFORMALITY OF A FUNCTION HOLOMORPHIC IN THE DISK

BY

SHINJI YAMASHITA

ABSTRACT. Conformality and semiconformality at a boundary point, of a function f nonconstant and holomorphic in $|z| < 1$ are local properties. Therefore one would suspect the requirement of such global conditions on f as f is univalent in $|z| < 1$, or f is a member of a larger class which contains all univalent functions in $|z| < 1$. We shall prove some extensions and new results without any assumption on f, or with a local assumption on f at most. Our methods are, for the most part, different from the ones in the classical cases. One of the main tools is Theorem 8 on the angular limits of the real part of a holomorphic function and its derivative.

1. Introduction. In the study of conformality and semiconformality at a boundary point, of a function f nonconstant and holomorphic in $|z| < 1$, one generally assumes global conditions, for example, f is schlicht in $|z| < 1$, f is a member of a linear-invariant normal family $\mathfrak{M}$, or equivalently, f is of class U_α for a certain $\alpha \geqslant 1$, etc. ([6], [8], [2]; for the definition of $\mathfrak{M}$ and U_α, see [7, p. 113 and p. 117]). Since conformality and semiconformality both are local properties, such global conditions as above seem superfluous. In the present study we shall obtain some extensions and new results without any assumption on f, or with a local assumption on f at most.

Let $D = \{|z| < 1\}$ and $\Gamma = \{|z| = 1\}$. By a sector at $\zeta \in \Gamma$ we mean a domain

$$S_\zeta(\alpha, r) = \left\{z;\, 0 < |z - \zeta| < r,\, \left|\arg(1 - \zeta^{-1}z)\right| < \alpha\right\}, \tag{1.1}$$

where $\alpha \in (0, \pi/2)$ and $r \in (0, \cos\alpha)$. By an admissible domain at ζ we mean a subdomain $\mathfrak{S}$ of D such that, for each $\alpha \in (0, \pi/2)$, there exists $r \in (0, \cos\alpha)$ with $S_\zeta(\alpha, r) \subset \mathfrak{S}$. Then $\mathfrak{S}$ contains a simply connected admissible domain $\mathfrak{S}_s$ at ζ.

Let $\mathbf{C}$ be the complex plane, and let Ω be the extended plane $\mathbf{C} \cup \{\infty\}$ with the chordal distance $\chi(\cdot,\cdot)$. Let $\mathfrak{S}$ be an admissible domain at $\zeta \in \Gamma$, let F be a mapping from $\mathfrak{S}$ into Ω, and let $\omega \in \Omega$. Then ω is called the angular limit of F at ζ if

$$\lim \chi(\omega, F(z)) = 0$$

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as $z \to \zeta$, $z \in S_\zeta(\alpha, r) \subset \mathfrak{S}$, for each $\alpha \in (0, \pi/2)$. A mapping F from D into Ω is said to have the radial limit $\omega \in \Omega$ at $\zeta \in \Gamma$ if

$$\lim \chi(\omega, F(t\zeta)) = 0$$

as $0 < t \nearrow 1$.

In this section we always assume that f is a nonconstant holomorphic function in D. Fix $\omega \in \mathbf{C}$ and $\zeta \in \Gamma$. The three meromorphic functions

$$\begin{aligned} f_{\zeta,\omega}(z) &= (\omega - f(z))/(\zeta - z), \\ \lambda(f)(z) &= f''(z)/f'(z), \\ f_\zeta(z) &= (\zeta - z)\lambda(f)(z) \end{aligned} \tag{1.2}$$

of z in D will play important roles. We begin with

THEOREM 1. *Let f be a function nonconstant and holomorphic in D. Let $c \in \mathbf{C}$ and $\zeta \in \Gamma$.*

(I) *Assume $c \neq 0$, and assume further that*

(1A) *there exists $\omega \in \mathbf{C}$ such that $f_{\zeta,\omega}/f'$ has the angular limit $-1/c$ at ζ. Then*

(1B) *f_ζ has the angular limit $c + 1$ at ζ.*

(II) *Assume that the real part* $\operatorname{Re} c < 0$. *Then* (1A), (1B) *and the following* ($1A_r$) *and* ($1A_a$) *are all equivalent.*

($1A_r$) *f has the radial limit $\omega \in \mathbf{C}$ at ζ, and $f_{\zeta,\omega}/f'$ has the angular limit $-1/c$ at ζ.*

($1A_a$) *f has the angular limit $\omega \in \mathbf{C}$ at ζ, and $f_{\zeta,\omega}/f'$ has the angular limit $-1/c$ at ζ.*

We call f semiconformal at $\zeta \in \Gamma$ [2, p. 74] if ($1A_r$) with $c = -1$ holds. Thus, f is semiconformal at ζ if and only if one of (1A), (1B), ($1A_a$) with $c = -1$, holds. D. M. Campbell and J. A. Pfaltzgraff [2, Theorem 4.3] proved that if $f \in U_\alpha$, and if

$$\lim_{|z|\to 1} (1 - |z|)|\lambda(f)(z)| = 0, \tag{1.3}$$

then f is semiconformal at each $\zeta \in \Gamma$. This is an immediate consequence of (II) of Theorem 1 with $c = -1$; in effect, $|\zeta - z|/(1 - |z|)$ (> 1) is bounded from above in each sector at ζ, whence (1B) with $c = -1$ holds at each $\zeta \in \Gamma$. We note that from the needless conditions: $f \in U_\alpha$, follows that

$$\sup_{z \in D} (1 - |z|)|\lambda(f)(z)| < +\infty; \tag{1.4}$$

actually, the supremum is bounded by $2(\alpha + 1)$. The example of $f \in U_\alpha$ for some α in [2, Theorem 4.3], which is semiconformal at each $\zeta \in \Gamma$, and which does not satisfy (1.3), is within the territory of Theorem 1; one can show that (1B) with $c = -1$ holds at each $\zeta \in \Gamma$.

We call f conformal at $\zeta \in \Gamma$ [6, p. 303] if f has the angular limit $\omega \in \mathbf{C}$ at ζ, and if $\arg f_{\zeta,\omega}$ has a finite angular limit at ζ. A few explanations must be given because f may not be schlicht in the whole D. There exists a simply connected admissible domain $\mathfrak{S}_s$ at ζ, in which we may consider each branch of $\arg f_{\zeta,\omega}$, and it has the finite angular limit at ζ. In the rest of the paper we omit the phrase "each branch of" if the situation is obvious. We now characterize the conformality in

THEOREM 2. *Let f be a function nonconstant and holomorphic in D, and let $\zeta \in \Gamma$. Then the following are equivalent.*

(2A) *f is conformal at ζ.*

(2B) $\arg f'$ *has a finite angular limit at ζ.*

(2C) *There exists $\omega \in \mathbf{C}$ such that* $\arg f_{\zeta,\omega}$ *has a finite angular limit at ζ.*

Again $\arg f'$ is considered in an $\mathfrak{S}_s$ at ζ; the same is true of $\arg f_{\zeta,\omega}$ described in (2C). Ch. Pommerenke [8, p. 257, Theorem 3.15, IV] proved the equivalence of (2B) and (2C) for $f \in \mathfrak{M}$, being again a superfluous condition. In [8, p. 257] and in [2, p. 74], (2C) with $\zeta = 1$ is adopted as the definition of conformality at $\zeta = 1$.

Now, f is said to have the angular derivative ω at $\zeta \in \Gamma$ if $\omega \in \mathbf{C}$ and if f' has the angular limit ω at ζ [6, p. 305]. Especially, (2B) is satisfied if f has a nonzero angular derivative at ζ. To consider the converse, we let $\mathfrak{C}(f)$ be the set of points $\zeta \in \Gamma$ at which f is conformal. Then we obtain

THEOREM 3. *Let f be a function nonconstant and holomorphic in D. Then f has a nonzero angular derivative at almost every point of $\mathfrak{C}(f)$.*

In other words, f' fails to have a nonzero angular limit at each point of the subset of $\mathfrak{C}(f)$ of outer Lebesgue measure zero. In effect, Theorem 3 follows from a more general result in §6. Especially, Theorem 3 in the case where f is schlicht in the whole D, is well known [6, p. 328, Corollary 10.4].

As to the relation of conformality and semiconformality we propose

THEOREM 4. *Let f be a function nonconstant and holomorphic in D. Assume that f is conformal at $\zeta \in \Gamma$. Then f is semiconformal at ζ.*

The converse is not true; for the proof, see the example of J. L. Walsh and D. Gaier [10, p. 85] with a few modifications.

In contrast with (1.4) we next consider a local condition at $\zeta \in \Gamma$:

$$\limsup |f_\zeta(z)| < +\infty \tag{1.5}$$

as $z \to \zeta$ in each sector at ζ. The condition (1.5) is equivalent to the condition:

$$\limsup (1 - |z|)|\lambda(f)(z)| < +\infty \tag{1.5'}$$

as $z \to \zeta$ in each sector at ζ. The set of points where (1.5) holds, is denoted by $X(f)$. If (1.4) holds, then $\Gamma = X(f)$, and if (1B) with $c \in \mathbf{C}$ holds, then $\zeta \in X(f)$.

By an admissible arc Λ at $\zeta \in \Gamma$ we mean a Jordan arc:

$$\Lambda: z = z(t) \in D, \quad t \in [0, 1), \quad \lim_{t \to 1} z(t) = \zeta, \tag{1.6}$$

such that there exist a sector S at ζ and $t_0 \in [0, 1)$ with $z(t) \in S$ for each $t \in [t_0, 1)$. A mapping F from an admissible Λ of (1.6) into Ω, is said to have the asymptotic value $\omega \in \Omega$ at ζ along Λ if

$$\lim_{t \to 1} \chi(\omega, F(z(t))) = 0.$$

For example, the radial limit at ζ is an asymptotic value at ζ.

THEOREM 5. *Let f be a function nonconstant and holomorphic in D, and let $\zeta \in X(f)$. Let $c \in \Omega$ in the present case. Then the condition* (1A) *is equivalent to the condition*:

(5A) *There exist $\omega \in \mathbf{C}$ and an admissible arc Λ at ζ along which $f_{\zeta,\omega}/f'$ has the asymptotic value $-1/c$.*

Furthermore, the condition (1B) *is equivalent to the condition*:

(5B) *There exists an admissible arc Λ at ζ along which f_ζ has the asymptotic value $c + 1$.*

We now consider the angular derivative at $\zeta \in X(f)$ in

THEOREM 6. *Let f be a function nonconstant and holomorphic in D, let $\zeta \in X(f)$, and let $a \in \mathbf{C}$. Assume that f has the angular limit $\omega \in \mathbf{C}$ at ζ.*

(I) *The following are equivalent.*

(6A) *There exists an admissible arc Λ at ζ along which f' has the asymptotic value a.*

(6B) *f has the angular derivative a at ζ.*

(II) *Assume that*

(6C) *for each $\alpha \in (0, \pi/2)$ there exists $r \in (0, \cos \alpha)$ such that $f - \omega$ never vanishes in $S_\zeta(\alpha, r)$.*

Then the following are equivalent.

(6D) *There exists an admissible arc Λ at ζ along which $f_{\zeta,\omega}$ has the asymptotic value a.*

(6E) *$f_{\zeta,\omega}$ has the angular limit a at ζ.*

(III) *Under the condition* (6C), *the four conditions* (6A), (6B), (6D), *and* (6E) *are equivalent.*

Theorem 6 is an angular analogue of the result [**6**, p. 305, Theorem 10.5] on schlicht functions; (6C) is satisfied if f is schlicht in an admissible domain at ζ.

Now, for f we define the meromorphic function, called the Schwarzian derivative:

$$\mathfrak{S}_f = \lambda(f)' - \tfrac{1}{2}\lambda(f)^2.$$

Let

$$\sigma(z, w) = \tfrac{1}{2}\log \frac{1 + \tau(z, w)}{1 - \tau(z, w)}, \qquad z, w \in D,$$

be the non-Euclidean distance, where

$$\tau(z, w) = |z - w|/|1 - \bar{z}w|$$

is again a distance in D (see [9, p. 510]). Let

$$H(z, \rho) = \{w \in D; \sigma(w, z) < \rho\}$$

be the non-Euclidean disk of center $z \in D$ and radius $\rho \in (0, +\infty]$. We now give criteria for $\zeta \in X(f)$ in

THEOREM 7. *Let f be a function nonconstant and holomorphic in D, and let $\zeta \in \Gamma$. Then the following are equivalent.*

(7A) $\zeta \in X(f)$.

(7B) $\limsup|\zeta - z|^2|\mathfrak{S}_f(z)| < +\infty$ *as $z \to \zeta$ in each sector at ζ.*

(7C) *For each $\alpha \in (0, \pi/2)$ there exist $r \in (0, \cos\alpha)$ and $\rho \in (0, +\infty]$ such that f is schlicht in each $H(z, \rho)$, $z \in S_\zeta(\alpha, r)$.*

2. A detailed study of angular limit. We consider the conformal homeomorphism Φ from D onto a given sector $S_1(\alpha, r)$ at 1 as follows. Let ψ be a conformal homeomorphism from D onto the half disk

$$\{z; 0 < |z| < r^{\pi/(2\alpha)}, |\arg z| < \pi/2\},$$

so that the extension of ψ to Γ sends 1 to 0. Then Φ is given by

$$\Phi = 1 - \psi^{2\alpha/\pi}; \tag{2.1}$$

the extension of Φ to Γ sends 1 to 1. For g holomorphic and bounded, $|g| < 1$, in D, we use the notation

$$g^\#(z) = (1 - |z|^2)|g'(z)|/(1 - |g(z)|^2), \qquad z \in D.$$

LEMMA 2.1. *For each $\beta \in (0, \alpha)$,*

$$\liminf \Phi^\#(w) > 0 \tag{2.2}$$

as $w \to 1$, $w \in \Phi^{-1}(S_1(\beta, r))$.

PROOF. Let $\tau(z)$ be the τ-distance of $z \in S_1(\beta, r/2)$ and the boundary $\partial S_1(\alpha, r) = \overline{S_1(\alpha, r)} - S_1(\alpha, r)$ of $S_1(\alpha, r)$ in the sense

$$\tau(z) = \inf\{\tau(w, z); w \in \partial S_1(\alpha, r)\} \qquad (\tau(1, z) = 1).$$

Then it is easy to see that

$$\liminf \tau(z) > 0 \tag{2.3}$$

as $z \to 1$, $z \in S_1(\beta, r/2)$ (for the proof, consult [9, p. 511, Theorem XI.2]). Now, for each fixed w, with $\Phi(w) \in S_1(\beta, r/2)$, we let F be the inverse of Φ in the τ-disk

$$D(\Phi(w)) = \{z \in D; \tau(z, \Phi(w)) < \tau(\Phi(w))\}.$$

Let

$$\Psi(z) = F\Big(\big(\tau(\Phi(w))z + \Phi(w)\big)/\big(1 + \bar{\Phi}(w)\tau(\Phi(w))z\big)\Big),$$

$z \in D$, and let

$$g(z) = (\Psi(z) - w)/\big(1 - \bar{w}\Psi(z)\big), \qquad z \in D.$$

Then the function g is bounded, $|g| < 1$, in D, with $g(0) = 0$. Schwarz' lemma now yields that

$$\tau(\Phi(w))/\Phi^{\#}(w) = |g'(0)| \leqslant 1,$$

whence follows (2.2) from (2.3).

THEOREM 8. *Let g be a function holomorphic in an admissible domain $\mathfrak{S}$ at $\zeta \in \Gamma$. Assume that the real part* $\operatorname{Re} g$ *has a finite angular limit at ζ. Then the function $(\zeta - z)g'(z)$ of $z \in \mathfrak{S}$ has the angular limit* 0 *at ζ.*

PROOF. It suffices to consider the case $\zeta = 1$. Let a be the angular limit of $h = \operatorname{Re} g$. Then, for each $\beta \in (0, \pi/2)$ we may find $\alpha \in (\beta, \pi/2)$ and $r \in (0, \cos\alpha)$ such that h may be extended continuously to the closure $\overline{S_1(\alpha, r)}$. Let Φ be as in (2.1), and let

$$F = g \circ \Phi - a \ \big(g \circ \Phi(w) = g(\Phi(w)), w \in D\big).$$

Then $H = \operatorname{Re} F = h \circ \Phi - a$ is continuous on $\bar{D}$, harmonic in D with $H(1) = 0$. Now, H may be expressed as the Poisson integral of $H(e^{it})$, $t \in [0, 2\pi]$. According to the lemma of A. Zygmund [11, p. 72, Lemma 6],

$$\big(1 - |w|^2\big)|F'(w)| \leqslant 2U(w), \qquad w \in D,$$

where U is the Poisson integral of the continuous function $U(e^{it}) = |H(e^{it})|$, $t \in [0, 2\pi]$. Since $U(1) = 0$, it follows from the well-known theorem [9, p. 130, Theorem IV.2] that

$$\lim_{\substack{w \to 1 \\ w \in D}} \big(1 - |w|^2\big)|F'(w)| = 0. \tag{2.4}$$

Now, for each $w \in \Phi^{-1}(S_1(\beta, r/2))$,

$$\big(1 - |\Phi(w)|^2\big)|g'(\Phi(w))| = \big(1 - |w|^2\big)|F'(w)|/\Phi^{\#}(w). \tag{2.5}$$

Combining (2.2), (2.4), and (2.5) we obtain

$$\lim\left(1-|z|^{2}\right)|g'(z)|=0$$

as $z \to 1$, $z \in S_1(\beta, r)$. Since β is arbitrary, and since $|1-z|/(1-|z|)$ is bounded in each $S_1(\beta, r/2)$, Theorem 8 follows.

PROOF OF (I) OF THEOREM 1. Assuming (1A) we set

$$h = f_{\zeta,\omega}/f' \tag{2.6}$$

in an admissible $\mathfrak{S}_s$ at ζ. A simple calculation yields

$$f_\zeta(z) = 1 - h(z)^{-1}\left[(\zeta - z)h'(z) + 1\right], \qquad z \in \mathfrak{S}_s. \tag{2.7}$$

Since h has the angular limit $-1/c$, $(\zeta - z)h'(z)$ has the angular limit 0 at ζ by Theorem 8. Therefore (1B) follows from (2.7).

Now, assuming (II) of Theorem 1, which will be proved in §4 by another method, we give

PROOF OF THEOREM 2. (2C) $\Rightarrow$ (2B). The real part of the function $H = -i\log f_{\zeta,\omega}$, defined in an $\mathfrak{S}_s$ at ζ, has a finite angular limit at ζ. It then follows from Theorem 8 that

$$h(z) = \left[1 - i(\zeta - z)H'(z)\right]^{-1}$$

has the angular limit 1 at ζ, where h is defined by (2.6). Therefore

$$\arg f' = \arg f_{\zeta,\omega} - \arg h$$

has a finite angular limit at ζ. Since (2A) $\Rightarrow$ (2C) is trivial, the rest we should prove is (2B) $\Rightarrow$ (2A). The assertion (2B), together with Theorem 8 for $i\log f'$, shows that if_ζ has the angular limit 0 at ζ. By (II) of Theorem 1 ((1A$_a$) with $c = -1$), f has the angular limit $\omega \in \mathbf{C}$ at ζ, and $f_{\zeta,\omega}/f'$ has the angular limit 1 at ζ. Therefore $\arg f_{\zeta,\omega}$ has a finite angular limit at ζ, so that (2A) holds.

Assuming (II) of Theorem 1 we also give

PROOF OF THEOREM 4. Since (2C) holds, it follows from the same argument as in the proof of Theorem 2 that h of (2.6) has the angular limit 1 at ζ. Theorem 4 now follows from (II) (1A$_r$), with $c = -1$, of Theorem 1.

Here we append

THEOREM 9. *Let f be a function nonconstant and holomorphic in D, let $\zeta \in \Gamma$, and let $c \in \mathbf{C}$ be arbitrary. Then from* (1B) *follows that*

(9A) $(\zeta - z)^2\mathfrak{S}_f(z)$ *has the angular limit* $\frac{1}{2}(1 - c^2)$ *at* ζ.

Thus, if f is semiconformal at ζ, then (1B) with $c = -1$ holds by Theorem 1, so that $(\zeta - z)^2\mathfrak{S}_f(z)$ has the angular limit 0 at ζ. However, the converse is false because $1 - c^2 = 0$ for $c = 1$ also.

PROOF OF THEOREM 9. After a short computation we have

$$(\zeta - z)^2\mathfrak{S}_f(z) = (\zeta - z)f_\zeta'(z) + f_\zeta(z) - \tfrac{1}{2}f_\zeta(z)^2,$$

which, combined with Theorem 8 for f_ζ, shows (9A).

3. Normal points. Let g be a function meromorphic in D, and let $\zeta \in \Gamma$. Then ζ is a normal point of g if

$$\limsup (1 - |z|) g^*(z) < +\infty \tag{3.1}$$

as $z \to \zeta$ in each sector at ζ, where

$$g^*(z) = |g'(z)| / \left(1 + |g(z)|^2\right).$$

Let $N(g)$ be the set of normal points of g.

LEMMA 3.1. *Let g be a function meromorphic in D, and let $\zeta \in N(g)$. Assume that g has the asymptotic value $\omega \in \Omega$ along an admissible arc at ζ. Then g has the angular limit ω at ζ.*

PROOF. It suffices to consider the case $\zeta = 1$. The lemma follows on proving the following:

Assume that g is meromorphic in a sector $S_1(\alpha, r)$ at 1 such that

$$\sup_{z \in S_1(\alpha, r)} (1 - |z|) g^*(z) < +\infty. \tag{3.2}$$

Let Λ be an admissible arc at 1 such that $\Lambda \subset S_1(\alpha, r)$. Assume that g has the asymptotic value $\omega \in \Omega$ at 1 along Λ. Then for each $\beta \in (0, \alpha)$,

$$\lim g(z) = \omega \tag{3.3}$$

as $z \to 1$, $z \in S_1(\beta, r)$.

For the proof we consider Φ of (2.1). Then a terminal part of $A = \Phi^{-1}(S_1(\beta, r))$, that is,

$$A \cap \{|z - 1| < \rho\}$$

for a $\rho \in (0, 1)$, is contained in a sector at 1. On the other hand, the composed function $F = g \circ \Phi$ satisfies

$$\sup_{z \in D} \left(1 - |z|^2\right) F^*(z) < +\infty$$

by (3.2) with $\Phi^\#(z) \leqslant 1$. Therefore F is normal in D in the sense of O. Lehto and K. I. Virtanen [**4**]. The assertion (3.3) now follows from the celebrated theorem [**4**, Theorem 2], applied to F, and from the described property of A.

LEMMA 3.2. *Let f be a function nonconstant and holomorphic in D. Then the following are true.*

$$X(f) \subset N(f'). \tag{3.4}$$

If $\zeta \in X(f)$ and if $\omega \in \mathbf{C}$ is arbitrary, then

$$\zeta \in N(f_\zeta) \cap N(f_{\zeta,\omega}/f'). \tag{3.5}$$

PROOF. Let $\zeta \in X(f)$. Since

$$f'^* < \tfrac{1}{2}|\lambda(f)|,$$

we obtain $\zeta \in N(f')$; this is (3.4). To prove $\zeta \in N(f_\zeta)$ of (3.5) we let $\beta \in (0, \pi/2)$ be arbitrary. Then there exist $\alpha \in (\beta, \pi/2)$ and $r \in (0, \cos\alpha)$ such that f_ζ is bounded in $S_\zeta(\alpha, r)$. Consider Φ of (2.1), and let

$$g(w) = f_\zeta(\zeta\Phi(w)), \qquad w \in D.$$

Since g is bounded in D, it is easy to observe that

$$\sup_{w \in D}\left(1 - |w|^2\right)|g'(w)| < +\infty. \tag{3.6}$$

Since

$$\left(1 - |w|^2\right)|\Phi'(w)| = \left(1 - |\zeta\Phi(w)|^2\right)\Phi^\#(w),$$

it follows that

$$\left(1 - |\zeta\Phi(w)|^2\right)|f_\zeta'(\zeta\Phi(w))| = \left(1 - |w|^2\right)|g'(w)|/\Phi^\#(w). \tag{3.7}$$

It then follows from Lemma 2.1 with (3.6) that the superior limit of the left-hand side of (3.7), as $w \to 1$, $w \in \Phi^{-1}(S_1(\beta, r))$, is bounded, which shows that $\zeta \in N(f_\zeta)$ because of $f_\zeta^* \leq |f_\zeta'|$. To prove $\zeta \in N(f_{\zeta,\omega}/f')$, we consider h of (2.6). Then

$$h'(z) = \frac{-(\zeta - z)f'(z)^2 - (\omega - f(z))\left[-f'(z) + (\zeta - z)f''(z)\right]}{(\zeta - z)^2 f'(z)^2},$$

which, combined with

$$|(\zeta - z)f'(z)|^2 + |\omega - f(z)|^2$$

$$\geq \max\left[|(\zeta - z)f'(z)|^2, 2|(\zeta - z)f'(z)(\omega - f(z))|\right],$$

shows that

$$h^*(z) \leq \frac{1}{|\zeta - z|} + \frac{1}{2}\,\frac{1}{|\zeta - z|} + \frac{1}{2}|\lambda(f)(z)|.$$

Therefore $\zeta \in N(h)$ if $\zeta \in X(f)$.

PROOF OF THEOREM 5 AND PROOF OF (I) OF THEOREM 6. Now follow immediately from Lemma 3.1 and Lemma 3.2.

4. An auxiliary function. The main aim of the present section is to prove

THEOREM 10. *Let f be a function nonconstant and holomorphic in D, let $\zeta \in \Gamma$ and let $c \in \mathbf{C}$ with* $\operatorname{Re} c < 0$. *Assume that*

(10A) *f_ζ has the radial limit $c + 1$ at ζ.*

Then

(10B) *f has the radial limit $\omega \in \mathbf{C}$, and $f_{\zeta,\omega}/f'$ has the radial limit $-1/c$ at ζ.*

Again in the present section f is a function nonconstant and holomorphic in D. For $\xi \in (0, 1)$ we consider an auxiliary function

$$f(z, \xi) = \frac{f(\varphi_\xi(z)) - f(\xi)}{(1 - \xi^2)f'(\xi)} \qquad (f'(\xi) \neq 0),$$

where

$$\varphi_\xi(z) = (z + \xi)/(1 + \xi z), \qquad z \in D.$$

Denoting by $f'(z, \xi)$ and $f''(z, \xi)$ the derivatives with respect to z, the function of z:

$$\begin{aligned} \lambda f(z, \xi) &= f''(z, \xi)/f'(z, \xi) \\ &= \lambda(f)(\varphi_\xi(z))\varphi_\xi'(z) + \lambda(\varphi_\xi)(z) \end{aligned} \tag{4.1}$$

is meromorphic in D under the condition that $f'(\xi) \neq 0$. For each $c \in \mathbf{C}$ we consider the generalized Koebe function

$$k_c(z) = \begin{cases} \dfrac{1}{2c}\left[\left(\dfrac{1+z}{1-z}\right)^c - 1\right], & c \neq 0, \\ \dfrac{1}{2}\log \dfrac{1+z}{1-z}, & c = 0, \end{cases}$$

in D. We now study (10A) for $\zeta = 1$ and $c \in \mathbf{C}$; thus,

(10A*) *f_1 has the radial limit $c + 1$ at 1.*

LEMMA 4.1. *Assume* (10A*). *Then there exists $x_0 \in (0, 1)$ such that for each $t \in (0, 1)$,*

$$f'(x, \xi) \to k_c'(x) \tag{4.2}$$

and

$$f(x, \xi) \to k_c(x) \tag{4.3}$$

as $x_0 < \xi \nearrow 1$ uniformly on $[0, t]$.

PROOF. Since (10A*) holds, there exist $x_0 \in (0, 1)$ and $M > 0$ such that

$$(1 - x^2)|\lambda(f)(x)| < M \tag{4.4}$$

and

$$f'(x) \neq 0 \tag{4.5}$$

for each $x \in [x_0, 1)$. Since $\varphi_\xi(x) \in [x_0, 1)$ for each $x \in [0, 1)$ and for each $\xi \in [x_0, 1)$, and since (4.1) holds, it follows that

$$\lambda f(x, \xi) = (1 - \varphi_\xi(x))\lambda(f)(\varphi_\xi(x)) \frac{1 + \xi}{(1 - x)(1 + \xi x)}$$
$$- \frac{2\xi}{1 + \xi x} \to \frac{2(x + c)}{1 - x^2} = \lambda(k_c)(x) \tag{4.6}$$

as $\xi \to 1$ uniformly on $[0, t]$; in effect, $\xi < \varphi_\xi(x) \nearrow 1$ as $\xi \to 1$. To show (4.2) we first prove that

$$f'(x, \xi)/k_c'(x) = \exp\left[\int_0^x \{\lambda f(y, \xi) - \lambda(k_c)(y)\}\, dy\right], \tag{4.7}$$

for each $x \in [0, t]$ and $\xi \in [x_0, 1)$, whence follows that $f'(x, \xi)/k_c'(x) \to 1$ as $\xi \to 1$ uniformly on $[0, t]$. In effect, for each fixed $\xi \in [x_0, 1)$, the holomorphic function $f'(z, \xi)$ of z never vanishes on $[0, t]$ by (4.5). Therefore there exists a simply connected domain $\mathfrak{D}_\xi \supset [0, t]$ where $f'(z, \xi)$ never vanishes. Consequently, $\log[f'(z, \xi)/k_c'(z)]$ is well defined in $\mathfrak{D}_\xi$, whose derivative is $\lambda f(z, \xi) - \lambda(k_c)(z)$ in $\mathfrak{D}_\xi$. We now obtain (4.7) by integration on $[0, x]$ $(x \in [0, t])$ because $f'(0, \xi) = k_c'(0) = 1$. Now k_c' is bounded on $[0, t]$, whence follows (4.2) from (4.7), combined with

$$|f'(x, \xi) - k_c'(x)| = |f'(x, \xi)/k_c'(x) - 1|\, |k_c'(x)|.$$

Since $f(0, \xi) = k_c(0) = 0$ $(\xi \in [x_0, 1))$, the similar argument shows (4.3).

Lemma 4.2. *Assume* (10A*), *and let* x_0 *be as in* (4.4) *and* (4.5). *Then, for each* $t \in (0, 1)$, *we may find a constant* $K(t) > 0$ *such that*

$$K(t)^{-1} \leqslant \frac{(1 - x)|f'(x, \xi)|}{(1 - y)|f'(y, \xi)|} \leqslant K(t) \tag{4.8}$$

for each $\xi \in [x_0, 1)$ *and for each pair* $x, y \in [0, 1)$ *with* $\tau(x, y) \leqslant t$.

Proof. Fix $\xi \in [x_0, 1)$, and consider the holomorphic function g of z in D defined by

$$g'(z) = (1 - z)f'(z, \xi), \quad g(0) = 0 \qquad (z \in D).$$

A direct calculation shows that

$$(1 - \eta^2)|\lambda(\varphi_\xi)(\eta)| \leqslant 4 \qquad (\eta \in [0, 1)).$$

It now follows from (4.1), together with $\varphi_\xi^\#(\eta) = 1$, that

$$\begin{aligned}&(1 - \eta^2)|\lambda(g)(\eta)|\\ &\quad \leqslant (1 + \eta) + (1 - \varphi_\xi(\eta)^2)|\lambda(f)(\varphi_\xi(\eta))| + (1 - \eta^2)|\lambda(\varphi_\xi)(\eta)|\\ &\quad \leqslant 6 + M = K\end{aligned} \tag{4.9}$$

for each $\eta \in [0, 1)$ because of (4.4) for $x = \varphi_\xi(\eta) \geqslant \xi \geqslant x_0$. Now, g' never vanishes on $[0, 1)$. Therefore, a simply connected domain $\mathfrak{D}_\xi^1 \supset [0, 1)$ exists, where g' never vanishes. Thus, $\log g'$ is well-defined in $\mathfrak{D}^1_\xi$, so that, by integrating $\lambda(g)(\eta)$ from x to y, we deduce from (4.9) that

$$|\log g'(y) - \log g'(x)| \leqslant K\sigma(x, y) \leqslant K\sigma(0, t) = K_1(t)$$

for x, y of (4.8). Now (4.8) follows on setting

$$K(t) = e^{K_1(t)} = \left(\frac{1+t}{1-t}\right)^{(6+M)/2},$$

being a constant independent of each $\xi \in [x_0, 1)$.

LEMMA 4.3. *Assume* (10A*) *with* $\gamma = -\operatorname{Re} c > 0$. *Then for each* $\delta \in (0, \gamma)$ *there exists* $x_\delta \in (0, 1)$ *such that*

$$|f'(x, \xi)| < K(1 - x)^{\delta - 1} \tag{4.10}$$

for each $x \in [0, 1)$ *and for each* $\xi \in [x_\delta, 1)$, *where* $K > 0$ *is a constant independent of* δ.

PROOF. We fix $t \in (0, 1)$ ($t = \frac{1}{2}$, say). Then

$$(1 - t^2)|k_c'(t)| = \left(\frac{1-t}{1+t}\right)^\gamma < \left(\frac{1-t}{1+t}\right)^\delta.$$

Since $f'(t, \xi) \to k_c'(t)$ as $\xi \to 1$ by Lemma 4.1, there exists $x_\delta \in [x_0, 1)$, depending on δ, such that

$$(1 - t^2)|f'(t, \xi)| < \left(\frac{1-t}{1+t}\right)^\delta \tag{4.11}$$

for each $\xi \in [x_\delta, 1)$. Set $\beta_0 = 0$, $\beta_{n+1} = \varphi_t(\beta_n)$ ($n \geqslant 0$). Then

$$\frac{1-\beta_n}{1+\beta_n} = \left(\frac{1-t}{1+t}\right)^n \qquad (n \geqslant 0) \tag{4.12}$$

by induction, so that $\beta_n \nearrow 1$ with $\tau(\beta_n, \beta_{n+1}) = t$ ($n \geqslant 0$). Now

$$\begin{aligned}
&(1 - \beta_{n+1}^2)|f'(\beta_{n+1}, \xi)| \\
&\quad = (1 - \beta_n^2)|f'(\beta_n, \xi)|(1 - t^2)|f'(t, \varphi_\xi(\beta_n))| \\
&\quad \leqslant \left(\frac{1-t}{1+t}\right)^\delta (1 - \beta_n^2)|f'(\beta_n, \xi)| \qquad (n \geqslant 1, \xi \in [x_\delta, 1))
\end{aligned}$$

because $\varphi_\xi(\beta_n) > \xi \geqslant x_\delta$ ($n \geqslant 1$). By induction with $\beta_1 = t$, and by (4.12) we conclude that

$$\begin{aligned}
(1 - \beta_n^2)|f'(\beta_n, \xi)| &\leqslant \left(\frac{1-t}{1+t}\right)^{n\delta} = \left(\frac{1-\beta_n}{1+\beta_n}\right)^\delta \\
&< (1 - \beta_n)^\delta \qquad (n \geqslant 1, \xi \in [x_\delta, 1)).
\end{aligned} \tag{4.13}$$

Now, for each $x \in [0, 1)$ there exists $n \geqslant 1$ such that $x \in [\beta_{n-1}, \beta_n]$. Since

$\tau(x, \beta_n) \leq t$, it follows from (4.8) and (4.13) for $\xi \in [x_\delta, 1)$ that

$$(1 - x)|f'(x, \xi)| \leq K(t)(1 - \beta_n)|f'(\beta_n, \xi)|$$
$$< K(t)(1 - \beta_n)^\delta \leq K(t)(1 - x)^\delta.$$

We thus obtain (4.10) with $K = K(t)$.

Proof of Theorem 10. It suffices to consider the case $\zeta = 1$. Set $\delta = -\frac{1}{2}\operatorname{Re} c$. It follows from (4.10) that the radial limit

$$f(1, \xi) = \lim_{0<x\to 1} f(x, \xi) \in \mathbf{C}$$

exists with

$$|f(1, \xi) - f(x, \xi)| \leq (K/\delta)(1 - x)^\delta \tag{4.14}$$

for each $x \in [0, 1)$ and for each $\xi \in [x_\delta, 1)$. Therefore

$$\left| \frac{f(1, \xi)(1 - \xi^2)f'(\xi) - f(\varphi_\xi(x)) + f(\xi)}{(1 - \xi^2)f'(\xi)} \right| \leq (K/\delta)(1 - x)^\delta$$

$$\big(x \in [0, 1), \xi \in [x_\delta, 1)\big). \tag{4.15}$$

Fixing ξ and letting $x \to 1$ in (4.15) we observe that the radial limit $\omega \in \mathbf{C}$ of f exists because $\varphi_\xi(x) \to 1$ as $x \to 1$. Furthermore,

$$f(1, \xi)(1 - \xi^2)f'(\xi) - \omega + f(\xi) = 0,$$

whence

$$f_{1,\omega}(\xi)/f'(\xi) = (1 + \xi)f(1, \xi) \qquad \big(\xi \in [x_\delta, 1)\big). \tag{4.16}$$

Returning to (4.14) we first fix $x = x_1$. We then assume that there exists a sequence $\xi_k \nearrow 1$ with $|f(1, \xi_k)| \to +\infty$. Letting $k \to \infty$ and considering Lemma 4.1, (4.3), in

$$|f(1, \xi_k)| \leq |f(x_1, \xi_k)| + (K/\delta)(1 - x_1)^\delta$$

we see that

$$+\infty \leq |k_c(x_1)| + (K/\delta)(1 - x_1)^\delta;$$

a contradiction. Now let $\beta \in \mathbf{C}$ be such that there exists a sequence $\eta_k \nearrow 1$ with $f(1, \eta_k) \to \beta$. Fixing $x \in [0, 1)$ and letting $k \to \infty$ in

$$|f(1, \eta_k) - f(x, \eta_k)| \leq (K/\delta)(1 - x)^\delta$$

we assert that

$$|\beta - k_c(x)| \leq (K/\delta)(1 - x)^\delta$$

again by Lemma 4.1, (4.3). Letting $x \to 1$ we observe that

$$\beta = \lim_{x\to 1} k_c(x) = -1/(2c).$$

Therefore

$$\lim_{\xi \to 1} f(1, \xi) = -1/(2c),$$

so that (4.16) shows (10B) with $\zeta = 1$.

PROOF OF (II) OF THEOREM 1. We have only to prove $(1\mathrm{B}) \Rightarrow (1\mathrm{A}_a)$, because $(1\mathrm{A}_a) \Rightarrow (1\mathrm{A}_r) \Rightarrow (1\mathrm{A})$ is trivial, and $(1\mathrm{A}) \Rightarrow (1\mathrm{B})$ follows from (I). Assume (1B) with $\operatorname{Re} c < 0$. Then by Theorem 10, the radial limit $\omega \in \mathbf{C}$ of f exists, and that $f_{\zeta,\omega}/f'$ has the radial limit $-1/c$ at ζ. Now, $\zeta \in X(f)$ because of (1B). Therefore $f_{\zeta,\omega}/f'$ has the angular limit $-1/c$ at ζ by Lemma 3.1 and Lemma 3.2. The rest we have to show is that ω is the angular limit of f. The function $g = \omega - f$ has the radial limit 0 at ζ, and

$$(\zeta - z)g'(z)/g(z) = -f'(z)/f_{\zeta,\omega}(z)$$

has the angular limit c at ζ. Thus, $\zeta \in N(g)$ because of $g^* \leq \frac{1}{2}|g'/g|$. Therefore g has the angular limit 0 at ζ by Lemma 3.1, whence ω is the angular limit of f.

We return to Lemma 4.3. For each $y \in [x_\delta, 1)$ we may find $x \in [0, 1)$ such that $y = \varphi_{x_\delta}(x)$. It then follows from (4.10) with $\varphi^{\#}_{x_\delta}(x) = 1$ and $1 - y \geq \frac{1}{2}(1 - x_\delta)(1 - x)$ that

$$\begin{aligned} |f'(y)| &\leq \frac{K(1 - x_\delta^2)|f'(x_\delta)|}{\varphi'_{x_\delta}(x)}(1 - x)^{\delta - 1} \\ &\leq K_2(1 - y)^{\delta - 1}, \end{aligned} \tag{4.17}$$

where K_2 is a constant depending on x_δ, δ and $f'(x_\delta)$, so that on δ and f. By integration we observe that

$$|\omega - f(y)| \leq (K_2/\delta)(1 - y)^{\delta}, \qquad y \in [x_\delta, 1), \tag{4.18}$$

where ω is the radial limit of (10B) with $\zeta = 1$.

Finally we propose

THEOREM 11. *Let f be a function nonconstant and holomorphic in D, and let $\zeta \in \Gamma$. Assume that* (1B) *with* $\operatorname{Re} c < -1$ *holds. Then f has the angular derivative* 0 *at ζ.*

PROOF. It suffices to consider the case $\zeta = 1$. Set $\delta = \frac{1}{2}(1 - \operatorname{Re} c)$. It then follows from (4.17) that f' has the radial limit 0 at 1. Theorem 11 follows from Lemma 3.1 and Lemma 3.2.

5. Characterizations of $X(f)$. The decisive result on the schlichtness of a function meromorphic in D in terms of the Schwarzian derivative is Z. Nehari's [5, Theorem I], which we express for the holomorphic case in

LEMMA 5.1. *Let φ be a function holomorphic and schlicht in D. Then*

$$\sup_{z \in D} \left(1 - |z|^2\right)^2 |\mathbb{S}_\varphi(z)| \leqslant 6. \tag{5.1}$$

Conversely if

$$\sup_{z \in D} \left(1 - |z|^2\right)^2 |\mathbb{S}_\varphi(z)| \leqslant 2 \tag{5.2}$$

for φ nonconstant and holomorphic in D, then φ is schlicht in D.

In contrast with Lemma 5.1, the result on the schlichtness of a function φ holomorphic in D in terms of $\lambda(\varphi)$ is the following, due to P. L. Duren, H. S. Shapiro and A. L. Shields [**3**, Theorem 2], and to J. Becker [**1**, p. 36, Corollary 4.1].

Lemma 5.2. *Let φ be a function holomorphic and schlicht in D. Then*

$$\sup_{z \in D} \left(1 - |z|^2\right) |\lambda(\varphi)(z)| \leqslant 6. \tag{5.3}$$

Conversely if

$$\sup_{z \in D} \left(1 - |z|^2\right) |\lambda(\varphi)(z)| \leqslant 1 \tag{5.4}$$

for φ nonconstant and holomorphic in D, then φ is schlicht in D.

The second half follows from Becker's cited result that if φ is holomorphic in D, $\varphi'(0) \neq 0$, and if

$$\sup_{z \in D} \left(1 - |z|^2\right) |z|\, |\lambda(\varphi)(z)| \leqslant 1,$$

then φ is schlicht in D.

Proof of Theorem 7. (7A) $\Rightarrow$ (7C). For each $\alpha \in [0, \pi/2)$ there exist $\beta \in (\alpha, \pi/2)$ and $r \in (0, \cos \beta)$ such that

$$\sup_{z \in S_\zeta(\beta, r)} \left(1 - |z|^2\right) |\lambda(f)(z)| = M < +\infty.$$

Then the τ-distance of each point of $S_\zeta(\alpha, r/2)$ to the boundary of $S_\zeta(\beta, r)$ is bounded from below by a constant $t_0 \in (0, 1)$. Choose $t \in (0, t_0)$ so small that

$$Mt/(1 - t^2) + 2t/(1 - t) \leqslant 1.$$

Fix $z \in S_\zeta(\alpha, r/2)$ and set

$$g = f \circ \varphi; \quad \varphi(w) = (tw + z)/(1 + t\bar{z}w), \quad w \in D. \tag{5.5}$$

Since

$$\lambda(g)(w) = \lambda(f)(\varphi(w))\varphi'(w) + \lambda(\varphi)(w)$$

it follows that

$$\begin{aligned}(1-|w|^2)&|\lambda(g)(w)|\\ &\leqslant (1-|\varphi(w)|^2)|\lambda(f)(\varphi(w))|\frac{t(1-|w|^2)}{1-t^2|w|^2}+\frac{2t|z|(1-|w|^2)}{|1+t\bar{z}w|}\\ &\leqslant Mt/(1-t^2)+2t/(1-t)\leqslant 1.\end{aligned}$$

Thus g is schlicht in D by Lemma 5.2, so that f is schlicht in $H(z, \rho)$ with $\rho = \sigma(0, t)$. Since $z \in S_\zeta(\alpha, r/2)$ is arbitrary (7C) follows.

(7C) $\Rightarrow$ (7A). Let

$$t = (e^{2\rho} - 1)/(e^{2\rho} + 1) \tag{5.6}$$

and consider g of (5.5) for each $z \in S_\zeta(\alpha, r)$ this time. We then apply Lemma 5.2 to g schlicht in D. Then $|\lambda(g)(0)| \leqslant 6$ by (5.3), whence

$$\left|t(1-|z|^2)\lambda(f)(z) - 2t\bar{z}\right| \leqslant 6$$

so that

$$\sup_{z\in S_\zeta(\alpha,r)}(1-|z|^2)|\lambda(f)(z)| \leqslant 2 + 6/t,$$

which shows that $\zeta \in X(f)$.

(7B) $\Rightarrow$ (7C). For each $\alpha \in (0, \pi/2)$ we choose β and r as in the proof of (7A) $\Rightarrow$ (7C) such that

$$\sup_{z\in S_\zeta(\beta,r)}(1-|z|^2)^2|\mathbb{S}_f(z)| = K < +\infty.$$

Then choose $t \in (0, t_0)$ so small that

$$Kt^2/(1-t^2)^2 \leqslant 2.$$

Fix $z \in S_\zeta(\alpha, r/2)$ and consider g of (5.5). Then

$$\mathbb{S}_g(w) = \mathbb{S}_f(\varphi(w))\varphi'(w)^2, \qquad w \in D,$$

because φ is linear. Therefore

$$\begin{aligned}(1-|w|^2)^2|\mathbb{S}_g(w)| &= (1-|\varphi(w)|^2)^2|\mathbb{S}_f(\varphi(w))|\frac{t^2(1-|w|^2)^2}{(1-t^2|w|^2)^2}\\ &\leqslant Kt^2/(1-t^2)^2 \leqslant 2.\end{aligned}$$

Consequently g is schlicht in D by Lemma 5.1, whence f is schlicht in each $H(z, \rho)$, $\rho = \sigma(0, t)$.

(7C) $\Rightarrow$ (7B). Let t be as in (5.6), and consider g of (5.5) for each $z \in S_\zeta(\alpha, r)$. Then

$$t^2\left(1-|z|^2\right)^2|\mathbb{S}_f(z)| = |\mathbb{S}_g(0)| \leqslant 6$$

by Lemma 5.1, so that

$$\sup_{z \in S_\zeta(\alpha,r)} \left(1-|z|^2\right)^2|\mathbb{S}_f(z)| \leqslant 6/t^2,$$

from which follows (7B).

LEMMA 5.3. *Let f be a function nonconstant and holomorphic in D, and let $\zeta \in X(f)$. Assume that* (6C) *holds for an $\omega \in \mathbf{C}$. Then $\zeta \in N(f_{\zeta,\omega})$.*

PROOF. First, (7C) is valid at $\zeta \in X(f)$. Then it follows from (6C) that (7C) for $g = 1/(f-\omega)$ is true. Therefore (7A) for g holds, that is, $\zeta \in X(g)$. Since

$$\lambda(g) = \lambda(f) - 2f'/(f-\omega),$$

it follows that

$$f'_{\zeta,\omega}(z)/f_{\zeta,\omega}(z) = 1/(\zeta - z) + \tfrac{1}{2}\left[\lambda(f)(z) - \lambda(g)(z)\right].$$

Combined with $f^*_{\zeta,\omega} \leqslant \frac{1}{2}|f'_{\zeta,\omega}/f_{\zeta,\omega}|$, the last equality shows that $\zeta \in N(f_{\zeta,\omega})$.

PROOF OF (II), (III) OF THEOREM 6. The equivalence of (6D) and (6E) follows from Lemma 3.1 and Lemma 5.3. To prove (III) it suffices to ascertain the equivalence of (6B) and (6E). Actually we can prove much more as the following theorem shows.

THEOREM 12. *Let f be a function nonconstant and holomorphic in D. Assume that f has the angular derivative $a \in \mathbf{C}$ at $\zeta \in \Gamma$. Then f has the angular limit $\omega \in \mathbf{C}$ at ζ such that $f_{\zeta,\omega}$ has the angular limit a at ζ. Conversely assume that there exists $\omega \in \mathbf{C}$ such that $f_{\zeta,\omega}$ has the angular limit $a \in \mathbf{C}$ at $\zeta \in \Gamma$. Then f has the angular limit ω and the angular derivative a at ζ.*

PROOF. First half. It is easy to see that $\zeta \in N(f)$, and that f has the radial limit ω at ζ. Therefore ω is the angular limit by Lemma 3.1. The first half immediately follows from

$$f_{\zeta,\omega}(z) = \int_0^1 f'(z + (\zeta - z)t)\,dt$$

on letting $z \to \zeta$ in each sector at ζ. To prove the second half we prepare

$$f'(z) = f_{\zeta,\omega}(z) - (\zeta - z)f'_{\zeta,\omega}(z), \qquad z \in D. \tag{5.7}$$

Letting $z \to \zeta$ in each sector at ζ, and considering Theorem 8 for $f_{\zeta,\omega}$, we deduce from (5.7) that f has the angular derivative a at ζ. Multiplying both sides of (5.7) by $(\zeta - z)$, we know that f has the angular limit ω at ζ.

6. An application of Plessner's theorem. Let f be a function nonconstant and holomorphic in D. Let $\mathfrak{B}(f)$ be the set of points $\zeta \in \Gamma$ such that there exists a sector at ζ, in wnich a certain branch of $\arg f'$ is bounded either from below or from above. If one branch of $\arg f'$ is bounded from below (above) in a sector at ζ, then the same is true of every branch of $\arg f'$. Since $\mathfrak{B}(f) \supset \mathfrak{C}(f)$, Theorem 3 now follows from

THEOREM 13. *Let f be a function nonconstant and holomorphic in D. Then f has a nonzero angular derivative at almost every point of $\mathfrak{B}(f)$.*

For the proof we need a lemma. Let $S = S_1(\alpha, r)$ be a sector at 1, and set

$$S_\zeta^* = \{z; \zeta^{-1}z \in S, q < |z| < 1\}, \tag{6.1}$$

where $q = (1 + r^2 - 2r\cos\alpha)^{1/2}$ and $\zeta \in \Gamma$.

LEMMA 6.1. *Each nonempty closed subset F of Γ may be decomposed as a finite union of nonempty closed sets F_ν $(1 \leqslant \nu \leqslant N)$ in the following sense:*

$$F = \bigcup_{\nu=1}^{N} F_\nu; \quad F_j \cap F_k = \varnothing \quad \text{if } j \neq k; \tag{6.2}$$

each open set

$$G_\nu = \bigcup_{\zeta \in F_\nu} S_\zeta^* \qquad (1 \leqslant \nu \leqslant N) \tag{6.3}$$

is a Jordan domain with the rectifiable boundary;

$$\bigcup_{\zeta \in F} S_\zeta^* = \bigcup_{\nu=1}^{N} G_\nu; \quad G_j \cap G_k = \varnothing \quad \text{if } j \neq k. \tag{6.4}$$

PROOF. Let a be the length of $S \cap \{|z| = q\}$. Then the boundary of each connected component of the left-hand side of (6.4) contains at least one arc on $\{|z| = q\}$ of length a. It is now easy to obtain F_ν's satisfying (6.2) and (6.4). The rectifiability of the boundary of each G_ν is proved by the familiar method (see, for example, [6, p. 324]).

PROOF OF THEOREM 13. Choose the sectors S_n $(n = 1, 2, \dots)$ at 1 such that each sector at 1 contains, and is contained in, respectively, one of S_n's. For simplicity we denote $S_{n\zeta}^* = (S_n)_\zeta^*$ of (6.1). Let A_n be the set of points $\zeta \in \Gamma$ such that f' never vanishes in $S_{n\zeta}^*$. Then A_n is closed $(n = 1, 2, \dots)$ on Γ, and

$$\mathfrak{B}(f) \subset \bigcup_{n=1}^{\infty} A_n. \tag{6.5}$$

To each $A_n \neq \varnothing$ we apply Lemma 6.1 with $S = S_n$. Then we obtain the decomposition

$$\bigcup_{k=1}^{k_n} A_{nk} = A_n, \tag{6.6}$$

described in (6.2), with

$$\bigcup_{\zeta \in A_n} S^*_{n\zeta} = \bigcup_{k=1}^{k_n} G_{nk},$$

where

$$G_{nk} = \bigcup_{\zeta \in A_{nk}} S^*_{n\zeta} \qquad (1 \leqslant k \leqslant k_n).$$

Fix one branch L_{nk} of $\log f'$ in G_{nk} $(1 \leqslant k \leqslant k_n)$. Let

$$A^+_{nkjm} = \{\zeta \in A_{nk};\ \operatorname{Im} L_{nk} \leqslant m \text{ in } S^*_{j\zeta}\},$$
$$A^-_{nkjm} = \{\zeta \in A_{nk};\ \operatorname{Im} L_{nk} \geqslant -m \text{ in } S^*_{j\zeta}\}, \tag{6.7}$$

where Im denotes the imaginary part, and $m = 1, 2, \ldots$ and j ranges over the set $J_n = \{j;\ S_j \subset S_n\}$. Then each A^+_{nkjm} and each A^-_{nkjm} together with the union

$$A_{nkjm} = A^+_{nkjm} \cup A^-_{nkjm}$$

are all closed on Γ. It is not difficult to observe that

$$\mathfrak{B}(f) \cap A_{nk} = \bigcup_{m=1}^{\infty} \bigcup_{j \in J_n} A_{nkjm}. \tag{6.8}$$

It now follows from (6.5), (6.6) and (6.8) that

$$\mathfrak{B}(f) = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{k_n} \bigcup_{m=1}^{\infty} \bigcup_{j \in J_n} A_{nkjm}. \tag{6.9}$$

Thus if L_{nk} has a finite angular limit at a.e. point of A_{nkjm} for each quartet n, k, j, m, then our theorem follows. Fix n, k, and let φ be a conformal homeomorphism from D onto G_{nk}, whose extension to Γ sends B_{nkjm} onto A_{nkjm}. Since the boundary ∂G_{nk} of G_{nk} is rectifiable, φ is conformal at a.e. point of Γ (see [6, p. 320, Theorem 10.11]). Furthermore we note that ∂G_{nk} and Γ have the common tangent at a.e. point of A_{nk}, whence at a.e. point of A_{nkjm}. Therefore a.e. point of B_{nkjm} is not a Plessner point [6, p. 323] of $g = L_{nk} \circ \varphi$ because of (6.7). It now follows from Plessner's theorem [6, p. 324, Theorem 10.13] that g has a finite angular limit at a.e. point of B_{nkjm}, whence the same is true of L_{nk} at a.e. point of A_{nkjm}.

REMARK. It is easy to see that $\mathfrak{B}(f) - \mathfrak{C}(f)$ is of measure zero. In effect, at a.e. point of $\mathfrak{B}(f)$, $\log f'$, whence $\arg f'$, have finite angular limits.

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DEPARTMENT OF MATHEMATICS, TOKYO METROPOLITAN UNIVERSITY, FUKAZAWA 2-1-1, SETAGAYA-KU, TOKYO, 158 JAPAN